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Duality-Induced Reflections and CPT

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The linear particle–antiparticle conjugation C and position space reflection P as well as the antilinear time reflection T are shown to be inducible by the self-duality of representations for the operation groups **SU**(2), **SL**($\mathbb{C}^2$), and $\mathbb{R}$ for spin, Lorentz transformations, and time translations, respectively. The definition of a color-compatible linear CP-reflection for quarks as self-duality induced is impossible since triplet and antitriplet **SU**(3)-representations are not linearly equivalent.

1. REFLECTIONS

1.1. Reflections

A reflection will be defined to be an involution of a finite-dimensional vector space V

$$V \overset{R}{\leftrightarrow} V, \qquad R \circ R = \mathrm{id}_V \Leftrightarrow R = R^{-1}$$

i.e., a realization of the parity group[2] $\mathbb{I}(2) = \{\pm 1\}$ in the V-bijections which is linear for a real space and may be linear or antilinear for a complex space

$$R(v + w) = R(v) + R(w), \qquad R(\alpha v) = \begin{cases} \alpha R(v) & \text{for } \alpha \in \mathbb{R} \text{ or } \mathbb{C} \quad \text{(linear)} \\ \overline{\alpha} R(v) & \text{for } \alpha \in \mathbb{C} \quad \text{(antilinear)} \end{cases}$$

An antilinear reflection for a complex space $V \cong \mathbb{C}^n$ is a real linear one for its real forms $V \cong \mathbb{R}^{2n}$.

The inversion of the real numbers $\alpha \leftrightarrow -\alpha$ is the simplest nontrivial linear reflection, and the canonical conjugation $\alpha \leftrightarrow \overline{\alpha}$ is the simplest nontrivial

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[2] Since the parity group is used as multiplicative group, I do not use the additive notation $\mathbb{Z}_2 = \{0, 1\}$.

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antilinear one, being a linear one of $\mathbb{C}$ considered as real 2-dimensional space $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$.

Any (anti)linear isomorphism $\iota: V \to W$ of two vector spaces defines an (anti)linear reflection of the direct sum

$$V \oplus W \overset{\iota \oplus \iota^{-1}}{\longleftrightarrow} V \oplus W$$

which will be denoted in brief also by $V \overset{\iota}{\leftrightarrow} W$.

1.2. Mirrors

The fixpoints of a linear reflection $V_R^+ = \{v \big| R(v) = v\}$, i.e., the elements with even parity, in an n-dimensional space constitute a vector subspace, the mirror for the reflection R, with dimension $0 \leq m \leq n$, with the complement $V_R^- = \{v \big| R(v) = -v\}$, i.e., the elements with odd parity, for the direct decomposition $V = V_R^+ \oplus V_R^-$. The central reflection $R = -\mathrm{id}_V$ has the origin as a 0-dimensional mirror. Linear reflections are diagonalizable,

$$R \cong \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & -\mathbf{1}_{n-m} \end{pmatrix}$$

with $(m, n - m)$ the signature characterizing the degeneracy of ± 1 in the spectrum of R. Conversely, any direct decomposition $V = V^+ \oplus V^-$ defines two reflections with the mirror either V^+ or V^-.

With $(\det R)^2 = 1$ any linear reflection has either a positive or a negative orientation. Looking in a 2-dimensional bathroom mirror is formalized by the negatively oriented 3-space reflection $(x, y, z) \leftrightarrow (-x, y, z)$. The position-space $\mathbb{R}^3$ reflection $\vec{x} \overset{-\mathbf{1}_3}{\leftrightarrow} -\vec{x}$ with negative orientation or the Minkowski spacetime translation $\mathbb{R}^4$ reflection $x \overset{-\mathbf{1}_4}{\leftrightarrow} -x$ with positive orientation are central reflections with the origins 'here' and 'here-now' as point mirrors. A space reflection $(x_0, \vec{x}) \overset{\mathrm{P}}{\leftrightarrow} (x_0, -\vec{x})$ in Minkowski space or a time reflection $(x_0, \vec{x}) \overset{\mathrm{T}}{\leftrightarrow} (-x_0, \vec{x})$ both have negative orientation with a 1-dimensional time and 3-dimensional position space mirror, respectively.

1.3. Reflections in Orthogonal Groups

A real linear reflection

$$R \cong \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & -\mathbf{1}_{n-m} \end{pmatrix}$$

can be considered to be an element of an orthogonal group $\mathbf{O}(p, q)$ for any[3]

[3] The orthogonal signature (p, q) has nothing to do with the reflection signature (n, m).

(p, q) with $p + q = n$. A positively oriented reflection, $\det R = 1$, is an element even of the special orthogonal groups, $R \in \mathbf{SO}(p, q)$, $p + q \geq 1$.

Orthogonal groups have discrete (semi)direct factor parity subgroups $\mathbb{I}(2)$, as seen in the simplest compact and noncompact examples

$$\mathbf{O}(2) \ni \begin{pmatrix} \epsilon \cos\alpha & \epsilon \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \qquad \epsilon \in \mathbb{I}(2) = \{\pm 1\}, \quad \alpha \in [0, 2\pi[$$

$$\mathbf{O}(1,1) \ni \epsilon' \begin{pmatrix} \epsilon \cosh\beta & \epsilon \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}, \qquad \epsilon, \epsilon' \in \mathbb{I}(2), \quad \beta \in \mathbb{R}$$

In general, the classes of a real orthogonal group with respect to its special normal subgroup constitute a reflection group

$$\mathbf{O}(p, q)/\mathbf{SO}(p, q) \cong \mathbb{I}(2)$$

For real, odd-dimensional spaces V, e.g., for position space $\mathbb{R}^3$, one has direct products of the special groups with the central reflection group, whereas for even-dimensional spaces, e.g., a Minkowski space $\mathbb{R}^4$, there arise semidirect products (denoted by $\overrightarrow{\times}$) of the special group with a reflection group which can be generated by any negatively oriented reflection,

$$\mathbf{O}(p, q) \cong \begin{cases} \mathbb{I}(2) \times \mathbf{SO}(p, q), & p + q = 1, 3, \ldots, \\ & \mathbb{I}(2) \cong \{\pm \mathrm{id}_V\} \\ \mathbb{I}(2) \overrightarrow{\times} \mathbf{SO}(p, q), & p + q = 2, 4, \ldots \\ & \mathbb{I}(2) \cong \{R, \mathrm{id}_V\} \text{ with } \det R = -1 \end{cases}$$

In the semidirect case the product is given as follows:

$$(I, \Lambda) \in \mathbb{I}(2) \overrightarrow{\times} \mathbf{SO}\ (p, q) \Rightarrow (I_1, \Lambda_1)\ (I_2, \Lambda_2) = (I_1 \circ I_2, \Lambda_1 \circ I_1 \circ \Lambda_2 \circ I_1)$$

Obviously, in the semidirect case the reflection group $\mathbb{I}(2)$ is not compatible with the action of the (special) orthogonal group,

$$p + q = 2, 4, \ldots, \qquad \det R = -1 \Rightarrow [R, \mathbf{SO}(p, q)] \neq \{0\}$$

For example, the group $\mathbf{O}(2)$ is nonabelian, or a space reflection and a time reflection of Minkowski space is not Lorentz group $\mathbf{SO}(1, 3)$-compatible.

For noncompact orthogonal groups there is another discrete reflection group: The connected subgroup G_0 (unit connection component and Lie algebra exponent) of a Lie group G is normal with a discrete quotient group G/G_0. The connected components of the full orthogonal groups are those of the special groups $\mathbf{O}_0(p, q) = \mathbf{SO}_0(p, q)$. For the compact case they are the special groups; for the noncompact ones, one has two components

$$\mathbf{SO}_0(n) = \mathbf{SO}(n)$$
$$pq \geq 1 \Rightarrow \quad \mathbf{SO}(p, q)/\mathbf{SO}_0(p, q) \cong \mathbb{I}(2)$$

Summarizing: A compact orthogonal group gives rise to a reflection group $\mathbb{I}(2)$,

$$\mathbf{O}(n) \cong \begin{cases} \{\pm\mathbf{1}_n\} \times \mathbf{SO}(n), & n = 1, 3, \ldots \\ \mathbb{I}(2) \overrightarrow{\times} \mathbf{SO}(n), & n = 2, 4, \ldots \end{cases}$$
$$\text{with} \quad \mathbb{I}(2) \cong \{R, \mathbf{1}_n\}, \qquad \det R = -1$$

a noncompact one to a reflection Klein group $\mathbb{I}(2) \times \mathbb{I}(2)$,

$$pq \geq 1\colon \mathbf{O}(p, q) \cong \begin{cases} \{\pm\mathbf{1}_{p+q}\} \times [\mathbb{I}(2) \overrightarrow{\times} \mathbf{SO}_0(p, q)], & p + q = 3, 5, \ldots \\ \mathbb{I}(2) \overrightarrow{\times} [\{\pm\mathbf{1}_{p+q}\} \times \mathbf{SO}_0(p, q)], & p + q = 2, 4, \ldots \end{cases}$$
$$\text{with} \quad \mathbb{I}(2) \cong \{R, \mathbf{1}_n\}, \qquad \det R = -1$$

For a noncompact $\mathbf{O}(p, q)$ with $p = 1$ the connected subgroup is the orthochronous group, compatible with the order on the vector space $V \cong \mathbb{R}^{1+q}$, e.g., for Minkowski spacetime

$$\mathbf{O}(1,3) \cong \mathbb{I}(2) \overrightarrow{\times} [\mathbb{I}(2) \times \mathbf{SO}_0(1, 3)]$$

where the reflection Klein group can be generated by the central reflection $-\mathbf{1}_4$ and a position-space reflection P

$$\mathbb{I}(2) \times \mathbb{I}(2) \cong \{P, \mathbf{1}_4\} \times \{\pm\mathbf{1}_4\} = \{\pm\mathbf{1}_4, \mathsf{P}, \mathsf{T} = -\mathsf{P}\},$$
$$[\mathbf{SO}_0(1, 3), \mathsf{P}] \neq \{0\}$$
$$P = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbf{1}_3 \end{pmatrix}, \qquad T = -\mathbf{1}_4 \circ P = \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{1}_3 \end{pmatrix}$$

Also, the connected subgroup $\mathbf{SO}_0(p, q)$ may contain positively oriented reflections, which are called continuous since they can be written as exponentials $R = e^l$ with an element of the orthogonal Lie algebra,[4] $l \in \log \mathbf{SO}_0(p, q)$, e.g., the central reflections $-\mathbf{1}_{2n} \in \mathbf{SO}(2n)$ in even-dimensional Euclidean spaces, e.g., in the Euclidean 2-plane. A negatively oriented reflection R of a space V can be embedded as a reflection $R \oplus S$ with any orientation of a strictly higher dimensional space $V \oplus W$,

$$V \overset{R}{\leftrightarrow} V, \qquad \det R = -1$$
$$V \oplus W \overset{R\oplus S}{\leftrightarrow} V \oplus W, \qquad \det(R \oplus S) = -\det S$$

[4] log G denotes the Lie algebra of the Lie group G.

where, for compact orthogonal groups on V and $V \oplus W$, a reflection $R \oplus S$ with det $S = -1$ is a continuous reflection, i.e., a rotation. There are familiar examples [2] for $\mathbf{O}(n) \hookrightarrow \mathbf{SO}(n+1)$: Two L-shaped noodles lying with opposite helicity on a kitchen table can be 3-space-rotated into each other, or a left- and a right-handed glove are identical up to Euclidean 4-space rotations. The embedding of the central position-space reflection into Minkowski spacetime can go into a positively or negatively oriented reflection which are both not continuous, i.e., they are in the discrete Klein reflection group

$$-\mathbf{1}_3 \hookrightarrow \begin{pmatrix} \pm 1 & 0 \\ 0 & -\mathbf{1}_3 \end{pmatrix}, \qquad \{P, -\mathbf{1}_4\} \subset \mathbf{O}(1,3)/\mathbf{SO}_0(1,3)$$

2. REFLECTIONS FOR SPINORS

The doubly connected groups $\mathbf{SO}(3)$ and $\mathbf{SO}_0(1, 3)$ can be complex represented via their simply connected covering groups $\mathbf{SU}(2)$ and $\mathbf{SL}(\mathbb{C}^2)$,[5] respectively,

$$\mathbf{SO}(3) \cong \mathbf{SU}(2)/\{\pm\mathbf{1}_2\}, \qquad \mathbf{SO}_0(1,3) \cong \mathbf{SL}(\mathbb{C}^2)/\{\pm\mathbf{1}_2\}$$

The reflection group $\{\pm\mathbf{1}_2\}$ for the $\mathbf{SO}(3)$-classes in $\mathbf{SU}(2)$ and the $\mathbf{SO}_0(1, 3)$-classes in $\mathbf{SL}(\mathbb{C}^2)$ contains the continuous central $\mathbb{C}^2$-reflection $-\mathbf{1}_2 = e^{i\pi\sigma_3} \in \mathbf{SU}(2)$.

2.1. The Pauli Spinor Reflection

The fundamental defining $\mathbf{SU}(2)$-representation for the rotations acts on Pauli spinors $W \cong \mathbb{C}^2$,

$$u = e^{i\vec{\alpha}\vec{\sigma}} \in \mathbf{SU}(2) \qquad \text{(Pauli matrices } \vec{\sigma}\text{)}$$

They have an invariant antisymmetric bilinear form (spinor 'metric')

$$\epsilon: \quad W \times W \to \mathbb{C}, \qquad \epsilon(\psi^A, \psi^B) = \epsilon^{AB} = -\epsilon^{BA}, \qquad A, B = 1, 2$$

which defines an isomorphism with the dual[6] space $W^T \cong \mathbb{C}^2$ which is compatible with the $\mathbf{SU}(2)$-action—on the dual space as dual representation $\check{u}$ (inverse transposed)

[5] Throughout this paper the group $\mathbf{SL}(\mathbb{C}^2)$ is used as *real* 6-dimensional Lie group.

[6] The linear forms V^T of a vector space V define the dual product $V^T \times V \to \mathbb{C}$ by $(\omega, v) = \omega(v)$ and dual bases by $\langle e_j, e^k\rangle = \delta_j^k$. Transposed mappings $f: V \to W$ are denoted by $f^T: W^T \to V^T$ with $\langle f^T(\omega), v\rangle = \langle \omega, f(v)\rangle$.

$$\begin{array}{ccc} W & \stackrel{u}{\to} & W \\ \epsilon\downarrow & & \downarrow\epsilon \\ W^T & \underset{\check{u}}{\to} & W^T \end{array}, \qquad \begin{array}{c} \psi^A \leftrightarrow \epsilon^{AB}\psi^*_B \\ \check{u} = u^{-1T} = u^{*-1} = \bar{u} = (e^{-i\vec{\alpha}\vec{\sigma}})^T \\ u = \epsilon^{-1} \circ \check{u} \circ \epsilon \\ -\vec{\sigma} = \epsilon^{-1} \circ \vec{\sigma}^T \circ \epsilon \end{array}$$

ϵ connects the two Pauli representations with reflected transformations of the spin Lie algebra log **SU**(2), i.e., it defines a central reflection for the three compact rotation parameters $\vec{\alpha}$,

$$e^{i\vec{\alpha}\vec{\sigma}} \stackrel{\epsilon}{\leftrightarrow} (e^{-i\vec{\alpha}\vec{\sigma}})^T$$

$$i\vec{\alpha}\vec{\sigma} \in \log \mathbf{SU}(2) \cong \mathbb{R}^3, \qquad \vec{\alpha} \stackrel{\epsilon}{\leftrightarrow} -\vec{\alpha}$$

and will be called the *Pauli spinor reflection*

$$W \stackrel{\epsilon}{\leftrightarrow} W^T, \qquad \psi^A \leftrightarrow \epsilon^{AB}\psi^*_B, \qquad [\epsilon, \mathbf{SU}(2)] = \{0\}$$

The mathematical structure of self-duality as a reflection generating mechnism is given in the Appendix.

2.2. Reflections C and P for Weyl Spinors

The two fundamental **SL** ($\mathbb{C}^2$)-representations for the Lorentz group are the the left- and right-handed Weyl representations on vector spaces W_L, $W_R \cong \mathbb{C}^2$ with the dual representations on the linear forms $W^T_{L,R}$,

$$\text{left:} \quad \lambda = e^{(i\vec{\alpha}+\vec{\beta})\vec{\alpha}} \in \mathbf{SL}(\mathbb{C}^2), \qquad \text{right:} \quad \hat{\lambda} = \lambda^{-1*} = e^{(i\vec{\alpha}-\vec{\beta})\vec{\sigma}}$$

$$\text{left dual:} \quad \check{\lambda} = \lambda^{-1T} = [e^{(-i\vec{\alpha}-\vec{\beta})\vec{\sigma}}]^T, \qquad \text{right dual:} \quad \lambda^{T*} = \bar{\lambda} = [e^{(-i\vec{\alpha}+\vec{\beta})\vec{\sigma}}]^T$$

The Weyl representations with dual bases in the conventional notations with dotted and undotted indices[7]

$$\text{left:} \quad l^A \in W_L \cong \mathbb{C}^2, \qquad \text{right:} \quad r^{\dot{A}} \in W_R \cong \mathbb{C}^2$$

$$\text{left dual:} \quad r^*_A \in W^T_L \cong \mathbb{C}^2, \qquad \text{right dual:} \quad l^*_{\dot{A}} \in W^T_R \cong \mathbb{C}^2$$

are self-dual with the **SL**($\mathbb{C}^2$)-invariant volume form on $\mathbb{C}^2$, i.e., the dual isomorphisms are Lorentz-compatible,

$$\begin{array}{ccc} W_L & \stackrel{\lambda}{\to} & W_R \\ \epsilon_k\downarrow & & \downarrow\epsilon_L \\ W^T_L & \underset{\check{\lambda}}{\to} & W^T_R \end{array}, \qquad \begin{array}{ccc} W_R & \stackrel{\hat{\lambda}}{\to} & W_R \\ \epsilon_R\downarrow & & \downarrow\epsilon_R \\ W^T_R & \underset{\bar{\lambda}}{\to} & W^T_R \end{array}$$

For the Lorentz group the spinor 'metric' will prove to be related to the

[7] The usual strange-looking crossover association of the letters l* and r* for right- and left-handed dual spinors, respectively, will be discussed later.

particle–antiparticle conjugation, and will be called *Weyl spinor reflection*, denoted by $\mathsf{C} \in \{\epsilon_L, \epsilon_R\}$,

$$W_L \overset{c}{\leftrightarrow} W_L^T, \qquad l^A \leftrightarrow \epsilon^{AB} r_B^*$$
$$W_R \overset{c}{\leftrightarrow} W_R^T, \qquad r^{\dot{A}} \leftrightarrow \epsilon^{\dot{A}\dot{B}} l_{\dot{B}}^*$$

There exist isomorphisms δ between left- and right-handed Weyl spinors, compatible with the spin group action, but not with the Lorentz group $\mathbf{SL}(\mathbb{C}^2)$,

$$\begin{array}{ccc} W_L & \overset{u_L}{\rightarrow} & W_L \\ \delta\downarrow & & \downarrow\delta \\ W_R & \underset{u_R}{\rightarrow} & W_R \end{array}, \qquad \begin{array}{l} u_{L,R} = e^{i\vec{\alpha}\vec{\sigma}} \in \mathbf{SU}(2) \\ l^A \leftrightarrow \delta_{\dot{A}}^{A} r^{\dot{A}} \end{array}$$

They connect representations with a reflected boost transformation, i.e., they define a central reflection for the three noncompact boost parameters $\vec{\beta}$,

$$e^{(i\vec{\alpha}+\vec{\beta})\vec{\sigma}} \overset{\delta}{\leftrightarrow} e^{(i\vec{\alpha}-\vec{\beta})\vec{\sigma}}$$

$$\vec{\sigma}\vec{\beta} \in \log \mathbf{SL}(\mathbb{C}^2)/\log \mathbf{SU}(2) \cong \mathbb{R}^3, \qquad \vec{\beta} \overset{\delta}{\leftrightarrow} -\vec{\beta}$$

These isomorphisms induce nontrivial reflections of the Dirac spinors $\Psi \in W_L \oplus W_R \cong \mathbb{C}^4$,

$$\Psi = \begin{pmatrix} l^A \\ r^{\dot{A}} \end{pmatrix} \overset{\delta}{\leftrightarrow} \begin{pmatrix} 0 & \delta_{\dot{B}}^{A} \\ \delta_{B}^{\dot{A}} & 0 \end{pmatrix} \begin{pmatrix} l^B \\ r^{\dot{B}} \end{pmatrix} = \gamma^0 \Psi$$

with the chiral representation of the Dirac matrices

$$\gamma^j = \begin{pmatrix} 0 & \check{\sigma}^j \\ \check{\sigma}^j & 0 \end{pmatrix}, \qquad \sigma^j = (\mathbf{1}_2, \vec{\sigma}), \qquad \check{\sigma}^j = (\mathbf{1}_2, -\vec{\sigma})$$

and will be called *Weyl spinor boost reflections* $\mathsf{P} = \delta$, later used for the central *position-space reflection* representation,

$$W_L \overset{\mathsf{P}}{\leftrightarrow} W_R, \qquad l^A \leftrightarrow \delta_{\dot{A}}^{A} r^{\dot{A}}$$
$$W_L^T \overset{\mathsf{P}}{\leftrightarrow} W_R^T, \qquad r_A^* \leftrightarrow \delta_{A}^{\dot{A}} l_{\dot{A}}^*$$

Therewith all four Weyl spinor spaces are connected to each other by linear reflections,

$$\begin{array}{ccc} W_L & \overset{\mathsf{P}}{\leftrightarrow} & W_R \\ \mathsf{C}\updownarrow & & \updownarrow\mathsf{C} \\ W_L^T & \underset{\mathsf{P}}{\leftrightarrow} & W_R^T \end{array}, \qquad \begin{array}{ll} [\mathsf{P}, \mathbf{SL}(\mathbb{C}^2)] \neq \{0\}, & [\mathsf{P}, \mathbf{SU}(2)] = \{0\} \\ [\mathsf{C}, \mathbf{SL}(\mathbb{C}^2)] = \{0\} & \end{array}$$

3. TIME REFLECTION

The time representations define the antilinear reflection T for time translation. The different duality with respect to $\mathbf{SL}(\mathbb{C}^2)$ and Lorentz group representations, on the one hand and time representations, on the other hand, leads to the nontrivial C, P, T cooperation.

3.1. Reflection T of Time Translations

The irreducible time representations, familiar from the quantum mechanical harmonic oscillator with time action eigenvalue (frequency) ω, with their duals (inverse transposed), are complex 1-dimensional,

$$t \mapsto e^{i\omega t} \in \mathbf{GL}(U), \qquad t \mapsto e^{-i\omega t} \in \mathbf{GL}(U^T), \qquad U \cong \mathbb{C} \cong U^T$$

They are self-dual (equivalent) with an antilinear dual isomorphism which is the $\mathbf{U}(1)$-conjugation for a dual basis $\mathrm{u} \in U$, $\mathrm{u}^* \in U^T$,

$$\begin{array}{ccc} U & \overset{e^{i\omega t}}{\rightarrow} & U \\ {*}\downarrow & & \downarrow{*} \\ U^T & \underset{e^{-i\omega t}}{\rightarrow} & U^T \end{array}, \qquad \mathrm{u} \leftrightarrow \mathrm{u}^*$$

The antilinear isomorphism $*$ defines a scalar product which gives rise to the quantum mechanical probability amplitudes (Fock state for the harmonic oscillator)

$$U \times U \to \mathbb{C}, \qquad \langle \mathrm{u}|\mathrm{u}\rangle = \langle \mathrm{u}^*, \mathrm{u}\rangle = 1$$

and defines the *time reflection* $T = *$ for the time translations

$$e^{i\omega t} \overset{\mathsf{T}}{\leftrightarrow} e^{-i\omega t}, \qquad t \overset{\mathsf{T}}{\leftrightarrow} -t$$

3.2. Lorentz Duality versus Time Duality

As anticipated in the conventional, on first sight strange-looking, dual Weyl spinor notation, e.g., $\mathrm{l} \in W_L$ and $\mathrm{l}^* \in W_R^T$, the Weyl spinor spaces $W_{L,R}^T$ with the dual left- and right-handed $\mathbf{SL}(\mathbb{C}^2)$-representations are not the spaces with the dual time representations as exemplified in the harmonic analysis of the left- and right-handed components in a Dirac field,

$$\mathrm{l}^A(x) = \int \frac{d^3q}{(2\pi)^3}\, s\left(\frac{q}{m}\right)^A_C \frac{e^{xiq}\mathrm{u}^C(\vec{q}) + e^{-xiq}a*^C(\vec{q})}{\sqrt{2}}$$

$$\mathrm{l}^*_{\dot{A}}(x) = \int \frac{d^3q}{(2\pi)^3}\, s^*\left(\frac{q}{m}\right)^C_{\dot{A}} \frac{e^{-xiq}u^*_C(\vec{q}) + e^{xiq}a_C(\vec{q})}{\sqrt{2}}$$

$$\mathrm{r}^{\dot{A}}(x) = \int \frac{d^3q}{(2\pi)^3}\, s^{*-1}\left(\frac{q}{m}\right)^{\dot{A}}_C \frac{e^{xiq}u^C(\vec{q}) - e^{-xiq}a*^C(\vec{q})}{\sqrt{2}}$$

$$\mathrm{r}^*_A(x) = \int \frac{d^3q}{(2\pi)^3}\, s^{-1}\left(\frac{q}{m}\right)^C_A \frac{e^{-xiq}u^*_C(\vec{q}) - e^{xiq}a_C(\vec{q})}{\sqrt{2}}$$

$$s\left(\frac{q}{m}\right) = \sqrt{\frac{q_0 + m}{2m}}\left(\mathbf{1} + \frac{\vec{\sigma}\vec{q}}{q_0 + m}\right), \qquad q = (q_0, \vec{q}), \qquad q_0 = \sqrt{m^2 + \vec{q}^2}$$

Here, $s(q/m) \in \mathbf{SL}(\mathbb{C}^2)$ is the Weyl representation of the boost from the rest system of the particle to a frame moving with velocity $\vec{q}/q_0$ (solution of the Dirac equation), u^C and a_C are the creation operators for particles and antiparticles with spin 1/2 and opposite charge number ± 1 and third spin direction, e.g., for electron and positron, u^*_C and $\mathrm{a}*^C$ are the corresponding annihilation operators.

Here, $*$ denotes the time representation dual $U \leftrightarrow U^*$, and T the Lorentz representation dual $W \leftrightarrow W^T$ (with spinor indices up and down), i:e., for the four types of Weyl spinors

$$\begin{array}{ccc} \mathrm{l}^A \in W_L & \overset{\text{time dual}}{\longleftrightarrow} & \mathrm{l}^*_{\dot{A}} \in W^T_R = W^*_L \\ \text{Lorentz dual}\updownarrow & & \updownarrow\text{Lorentz dual} \\ \mathrm{r}^*_A \in W^T_L = W^*_R & \underset{\text{time dual}}{\longleftrightarrow} & \mathrm{r}^{\dot{A}} \in W_R \end{array}$$

Time representation duality does not coincide with Lorentz group representation duality.

The antilinear time reflection [$\mathbf{U}(1)$-conjugation] $T = *$ is compatible with the action of the little group $\mathbf{SU}(2)$, not with the full Lorentz group,

$$\left.\begin{array}{l} W_L \overset{\mathsf{T}}{\leftrightarrow} W^T_R,\ \mathrm{l}^A \leftrightarrow \delta^{A\dot{A}}\mathrm{l}^*_{\dot{A}} \\ W_R \overset{\mathsf{T}}{\leftrightarrow} W^T_L,\ \mathrm{r}^{\dot{A}} \leftrightarrow \delta^{\dot{A}A}\mathrm{r}^*_A \end{array}\right\}, \qquad [T, \mathbf{SL}(\mathbb{C}^2)] \neq 0, \qquad [T, \mathbf{SU}(2)] = 0$$

3.3. The Cooperation of C, P, T in the Lorentz Group

It is useful to summarize the action of the linear Weyl spinor reflections C (particle–antiparticle conjugation) and P (position-space central reflection) and the antilinear time reflection T in the two types of commuting diagrams

$$\begin{array}{ccc} W_L & \overset{\mathsf{P}}{\leftrightarrow} & W_R \\ \mathsf{C}\updownarrow & & \updownarrow\mathsf{C} \\ W_L^T & \underset{\mathsf{P}}{\leftrightarrow} & W_R^T \end{array} \quad \text{with} \quad \begin{array}{ccc} \mathrm{l}^A & \overset{\mathsf{P}}{\leftrightarrow} & \delta_A^A \mathrm{r}^{\dot{A}} \\ \mathsf{C}\updownarrow & & \updownarrow\mathsf{C} \\ \epsilon^{AB}\mathrm{r}_B^* & \underset{\mathsf{P}}{\leftrightarrow} & \delta^{A^A}\epsilon^{A\dot{B}}\mathrm{l}_B^* \end{array}$$

$$\begin{array}{ccc} W_L & \overset{\mathsf{T}}{\leftrightarrow} & W_R^T \\ \mathsf{C}\updownarrow & & \updownarrow\mathsf{C} \\ W_L^T & \underset{\mathsf{T}}{\leftrightarrow} & W_R \end{array} \quad \text{with} \quad \begin{array}{ccc} \mathrm{l}^A & \overset{\mathsf{T}}{\leftrightarrow} & \delta^{A\dot{B}}\mathrm{l}_B^* \\ \mathsf{C}\updownarrow & & \updownarrow\mathsf{C} \\ \epsilon^{AB}\mathrm{r}_B^* & \underset{\mathsf{T}}{\leftrightarrow} & \delta^{A\dot{B}}\epsilon_{\dot{B}\dot{A}}\mathrm{r}^{\dot{A}} \end{array}$$

with the compatibilities

$$[\mathsf{C}, \mathbf{SL}(\mathbb{C}^2)] = \{0\}, \qquad [\mathsf{P} \text{ and } \mathsf{T}, \mathbf{SL}(\mathbb{C}^2)] \neq \{0\}, \qquad [\mathsf{P} \text{ and } \mathsf{T}, \mathbf{SU}(2)] = \{0\}$$
$$[\mathsf{C}, \mathsf{P}] = 0, \qquad [\mathsf{C}, \mathsf{T}] = 0, \qquad [\mathsf{P}, \mathsf{T}] = 0$$

The product CPT is an antilinear reflection of each Weyl spinor space, e.g., for the left-handed spinors

$$W_L \overset{\mathsf{CPT}}{\leftrightarrow} W_L, \qquad \mathrm{l}^A \leftrightarrow \delta_A^{\dot{A}}\epsilon^{\dot{A}\dot{B}}\delta_{\dot{B}B}\mathrm{l}^B$$

involving an element of the group $\mathbf{SL}(\mathbb{C}^2)$, even of $\mathbf{SU}(2)$,

$$\mathsf{CPT} \sim \delta_{\dot{A}}^A\epsilon^{\dot{A}\dot{B}}\delta_{\dot{B}B} = u_B^A \cong \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = e^{i\pi\sigma_2/2} \in \mathbf{SU}(2) \subset \mathbf{SL}(\mathbb{C}^2)$$

This element gives, in the used basis, for the Lorentz group a π-rotation around the second axis in position space, i.e., a continuous reflection

$$\mathbf{SU}(2) \ni e^{i\pi\sigma_2/2} \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathbf{SO}(3), \qquad (x, y, z) \leftrightarrow (-x, y, -z)$$

The fact that the antilinear CPT-reflection is, up to a number conjugation (indicated by overbar), an element of $\mathbf{SL}(\mathbb{C}^2)$, covering the connected Lorentz group $\mathbf{SO}_0(1,3)$, is decisive for the proof of the well-known CPT-theorem[4, 3]

$$\overline{\mathsf{CPT}} \in \mathbf{SL}(\mathbb{C}^2)$$

4. SPINOR-INDUCED REFLECTIONS

The linear spinor reflections ϵ for Pauli spinors and C, P for Weyl spinors are inducible on all irreducible finite-dimensional representations of $\mathbf{SU}(2)$ and $\mathbf{SL}(\mathbb{C}^2)$ with their adjoint groups $\mathbf{SO}(3)$ and $\mathbf{SO}_0(1, 3)$, respectively, via the general procedure: Given the group G action on two vector spaces, its tensor product representation reads

$$G \times (V_1 \otimes V_2) \to V_1 \otimes V_2, \qquad g \bullet (v_1 \otimes v_2) = (g \bullet v_1) \otimes (g \bullet v_2)$$

A realization of the simple reflection group $\mathbb{I}(2) = \{\pm 1\}$ is either faithful or trivial.

4.1. Spinor-Induced Reflection of Position Space

The reflection $W \overset{\epsilon}{\leftrightarrow} W^T$ for a Pauli spinor space $W \cong \mathbb{C}^2$ induces the central reflection of position space whose elements come, in the Pauli representation of position space, as traceless hermitian (2×2) matrices

$$\vec{x}: \quad W \to W, \qquad \mathrm{tr}\,\vec{x} = 0, \qquad \vec{x} = \vec{x}^* = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}$$

i.e., as elements[8] of the tensor product $W \otimes W^T$ with the induced $\boldsymbol{\epsilon}$-reflection

$$-\vec{\sigma} = \boldsymbol{\epsilon}_0^{-1}\vec{\boldsymbol{\sigma}}_0^T\boldsymbol{\epsilon} \Rightarrow \vec{x} \overset{\epsilon}{\leftrightarrow} \boldsymbol{\epsilon}_0^{-1}\vec{x}_0^T\boldsymbol{\epsilon} = -\vec{x}$$

In the Cartan representation the Minkowski spacetime translations are hermitian mappings from right-handed to left-handed spinors,

$$x: \quad W_R \to W_L, \qquad x = x^* = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$$

i.e., tensors in the product $W_L \otimes W_R^T$. The linear CP-reflection for Weyl spinors

$$W_L \overset{\mathsf{CP}}{\leftrightarrow} W_R^T, \qquad W_R \overset{\mathsf{CP}}{\leftrightarrow} W_L^T$$

induces the position-space reflection of Minkowski spacetime,

$$\sigma^j = (\mathbf{1}_2, \vec{\sigma}), \qquad \boldsymbol{\epsilon}^{-1} \circ (\sigma^j)^T \circ \boldsymbol{\epsilon} = \sigma_j = (\mathbf{1}_2 - \vec{\sigma})$$

$$x \cong (x_0, \vec{x}) \overset{\mathsf{CP}}{\leftrightarrow} \boldsymbol{\epsilon}^{-1} \circ x^T \circ \boldsymbol{\epsilon} = \begin{pmatrix} x_0 - x_3 & -x_1 + ix_2 \\ -x_1 - ix_2 & x_0 + x_3 \end{pmatrix} \cong (x_0, -\vec{x})$$

4.2. Induced Reflections of Spin Representation Spaces

All irreducible complex representations of the spin group **SU**(2) with $2J = 0, 1, 2, \ldots$ have an invariant bilinear form arising as a symmetric

[8] The linear mappings $\{V \to W\}$ for finite-dimensional vector spaces are naturally isomorphic to the tensor product $W \otimes V^T$ with the linear V-formsV^T.

tensor product of the antisymmetric spinor 'metric' ϵ. The bilinear form is given for the irreducible representation $[2J] \cong \vee^{2J} u$ on the vector space $\vee^{2J} W \cong \mathbb{C}^{2J+1}$ by the corresponding totally symmetric[9] power and is antisymmetric for half-integer spin and symmetric for integer spin,

$$\epsilon^{2J} = \bigvee^{2J} \epsilon, \qquad \epsilon^{2J}(v, w) = \begin{cases} +\epsilon^{2J}(w, v), & 2J = 0, 2, 4, \ldots \\ -\epsilon^{2J}(w, v), & 2J = 1, 3, \ldots \end{cases}$$

The complex representation spaces for integer spin $J = 0, 1, \ldots$, acted upon faithfully only with the special rotations $\mathbf{SO}(3) \cong \mathbf{SU}(2)/\{\pm \mathbf{1}_2\}$, are direct sums of two irreducible real $\mathbf{SO}(3)$-representation spaces $\mathbb{R}^{2J+1}$ where the invariant bilinear form is symmetric and definite, e.g., the negative-definite Killing form $-\mathbf{1}_3$ for the adjoint representation $[2] \cong u \vee u$ on $\mathbb{R}^3$.

The Pauli spinor reflection induces the reflections for the irreducible spin representation spaces

$$V \cong \bigvee^{2J} W \cong \mathbb{C}^{2J+1}: \qquad V \overset{\epsilon^{2J}}{\leftrightarrow} V^T$$

For integer spin (odd-dimensional representation spaces) the two real subspaces with irreducible real $\mathbf{SO}(3)$-representation come with a trivial $-\mathbf{1}_3 \mapsto \mathbf{1}_{2J+1}$ and a faithful $-\mathbf{1}_3 \mapsto -\mathbf{1}_{2J+1} \in \mathbf{O}(2J + 1)/\mathbf{SO}(2J + 1)$ representation of the central position-space reflection, as seen in the diagonalization of the induced reflection

$$\begin{pmatrix} 0 & \epsilon^{2J} \\ [\epsilon^{2J}]^{-1} & 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & J = 0 \\ \begin{pmatrix} 0 & \epsilon \\ \epsilon^{-1} & 0 \end{pmatrix}, & J = \frac{1}{2} \\ \begin{pmatrix} 0 & -\mathbf{1}_3 \\ -\mathbf{1}_3 & 0 \end{pmatrix} \cong \begin{pmatrix} \mathbf{1}_3 & 0 \\ 0 & -\mathbf{1}_3 \end{pmatrix}, & J = 1 \\ \text{etc.} \end{cases}$$

The decomposition for the integer spin representation spaces uses symmetric and antisymmetric tensor products, as illustrated for the scalar and vector spin representation with a Pauli spinor basis,

[9] $\bigvee$ and $\bigwedge$ denote symmetrized and antisymmetrized tensor products.

$$W \overset{\epsilon}{\leftrightarrow} W^T, \qquad \psi^A \leftrightarrow \epsilon^{AB}\psi^*_B, \qquad J = \frac{1}{2}$$

$$W^T \otimes W \overset{\epsilon}{\leftrightarrow} W \otimes W^T, \qquad \begin{cases} \psi^*_A \otimes \psi^A \leftrightarrow \psi^A \otimes \psi^*_A, & J = 0 \\ \vec{\sigma}^A_B\, \psi^*_A \otimes \psi^B \leftrightarrow -\, \vec{\sigma}^A_B\, \psi^B \otimes \psi^*_A, & J = 1 \end{cases}$$

Writing for the tensor (anti)commutator $[a, b]_\epsilon = a \otimes b + \epsilon b \otimes a$ with $\epsilon = \pm 1$, one has in both cases one trivial and one faithful reflection representation,

$$[\psi^*_A, \psi^A]_\epsilon \leftrightarrow \epsilon[\psi^*_A, \psi^A]_\epsilon \qquad J = 0$$

$$[\psi^*_A\vec{\sigma}^A_B, \psi^B]_\epsilon \leftrightarrow -\epsilon[\psi^*_A\vec{\sigma}^A_B, \psi^B]_\epsilon, \qquad J = 1$$

4.3. Induced Reflections of Lorentz Group Representation Spaces

The generating structure of the two Weyl representations induces C, P-reflections of $\mathbf{SL}(\mathbb{C}^2)$-representations spaces.

The complex finite-dimensional irreducible representations of the group $\mathbf{SL}(\mathbb{C}^2)$ are characterized by two spins $[2L|2R]$ with integer and half-integer $L, R = 0, 1/2, 1, \ldots$. They are equivalent to the totally symmetric products of the left- and right-handed Weyl representations

$$\text{Weyl left:} \quad [1|0] = \lambda = e^{(i\vec{\alpha}+\vec{\beta})\vec{\sigma}}, \qquad \text{Weyl right:} \quad [0|1] = \hat{\lambda} = e^{(i\vec{\alpha}-\vec{\beta})\vec{\sigma}}$$

$$[2L|2R] \cong \overset{2L}{\bigvee} \lambda \otimes \overset{2R}{\bigvee} \hat{\lambda} \quad \text{acting on} \quad V \cong \overset{2L}{\bigvee} W_L \otimes \overset{2R}{\bigvee} W_R \cong \mathbb{C}^{(2L+1)(2R+1)}$$

$[2L|2R]$ and $[2R|2L]$ are equivalent with respect to the subgroup $\mathbf{SU}(2)$-representations. The induced reflections are given by the corresponding products of the Weyl spinor reflections.

The real representation spaces for the Lorentz group $\mathbf{SO}_0(1, 3)$ are characterized by integer spin

$$L + R = 0, 1, 2, \ldots$$

They are all generated by the Minkowski representation $[1|1] \cong \lambda \otimes \bar{\lambda}$, where the complex 4-dimensional representation space is decomposable into two real 4-dimensional ones, a hermitian and an antihermitian tensor,

$$\mathbb{C}^4 \cong W_L \otimes W_R^T \ni \mathrm{l} \otimes \mathrm{l}^* = z = x + i\alpha \in \mathbb{R}^4 \otimes i\mathbb{R}^4$$

With Weyl spinor bases the induced linear reflections for the Minkowski representation look as follows [with $\sigma^j = (\mathbf{1}_2, \vec{\sigma}) = \check{\sigma}_j$ and $\sigma_j = (\mathbf{1}_2, -\vec{\sigma}) = \check{\sigma}^j$]:

$$\sigma^j \overset{\mathsf{P}}{\leftrightarrow} \check{\sigma}^T_j, \qquad \mathrm{l}^*\sigma^j\mathrm{l} \overset{\mathsf{P}}{\leftrightarrow} \mathrm{r}^*\check{\sigma}^T_j\,\mathrm{r}$$

$$\sigma_j \overset{\mathsf{C}}{\leftrightarrow} \check{\sigma}^T_j, \qquad \mathrm{l}^*\sigma_j\,\mathrm{l} \overset{\mathsf{C}}{\leftrightarrow} \mathrm{r}\,\check{\sigma}^T_j\,\mathrm{r}^*$$

$$\sigma^j \overset{\mathrm{CP}}{\leftrightarrow} \sigma_j^T, \qquad l^*\sigma^j l \overset{\mathrm{CP}}{\leftrightarrow} l\sigma_j^T l^*, \qquad r^*\check{\sigma}^j r \overset{\mathrm{CP}}{\leftrightarrow} r\check{\sigma}_j^T r^*$$

and can be arranged in combinations of definite parity, e.g., for P with Dirac spinors in a vector $\overline{\Psi}\gamma^j\Psi$ and an axial vector $\overline{\Psi}\gamma^j\gamma_5\Psi$. The antilinear time reflection has to change in addition the order in the product,

$$\sigma^j \overset{\mathrm{T}}{\leftrightarrow} \sigma_j, \qquad l^*\sigma^j l \overset{\mathrm{T}}{\leftrightarrow} l^*\sigma_j l, \qquad r^*\check{\sigma}^j r \overset{\mathrm{T}}{\leftrightarrow} r^*\check{\sigma}_j r$$

4.4. Reflections of Spacetime Fields

A field Φ is a mapping from position space $\mathbb{R}^3$ or, as relativistic field, from Minkowski spacetime $\mathbb{R}^4$ with values in a complex vector space V with the action of a group G both on space(time) and on V. This defines the action of the group on the field $\Phi \mapsto g \bullet \Phi = {}_g\Phi$ by the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{R}^3, \mathbb{R}^4 & \overset{O(g)}{\rightarrow} & \mathbb{R}^3, \mathbb{R}^4 \\ {}^{\Phi}\downarrow & & \downarrow {}_g\Phi \\ V & \underset{D(g)}{\rightarrow} & V \end{array}, \qquad {}_g\Phi(x) = D(g)\Phi(O(g^{-1}).x) \quad \text{for} \quad g \in G$$

For position space the external action group is the Euclidean group $\mathbf{O}(3) \vec{\times} \mathbb{R}^3$, for Minkowski spacetime the Poincaré group $\mathbf{O}(1, 3) \vec{\times} \mathbb{R}^4$. The value space may have additional integral action groups, e.g., $\mathbf{U}(1)$, $\mathbf{SU}(2)$, and $\mathbf{SU}(3)$ hypercharge, isospin, and color, respectively in the standard model for quark and lepton fields.

For Pauli spinor fields on position space the $\mathbf{O}(3)$-action has a direct $\mathbf{SU}(2)$-factor and a reflection factor $\mathbb{I}(2)$,

$$\psi: \quad \mathbb{R}^3 \to W \cong \mathbb{C}^2,$$
$$\begin{cases} {}_u\psi(\vec{x}) = D(u)\psi\,(O(u^{-1}).\vec{x}), & u \in \mathbf{SU}(2),\ O(u) \in \mathbf{SO}(3) \\ \psi^A(\vec{x}) \overset{\epsilon}{\leftrightarrow} \epsilon^{AB}\psi_B^*(-\vec{x}), & \text{position reflection } \mathbb{I}(2) \end{cases}$$

Spacetime fields have the Lorentz group behavior

$${}_\lambda\Phi(x) = D(\lambda).\Phi(O(\lambda^{-1}).x), \qquad \lambda \in \mathbf{SL}(\mathbb{C}^2), \qquad O(\lambda) \in \mathbf{SO}_0(1, 3)$$

The antilinear time reflection uses the conjugation to the time dual field

$$\Phi(x_0, \vec{x}) \overset{T}{\leftrightarrow} \Phi^*(-x_0, \vec{x})$$

The reflections for Weyl spinor fields on Minkowski spacetime are

$$1^A(x_0, \vec{x}) \overset{\mathsf{P}}{\leftrightarrow} \delta^A_{\dot{A}} r^{\dot{A}} \quad (x_0, -\vec{x})$$

$$(l^A, r^{\dot{A}}) \quad (x_0, \vec{x}) \overset{\mathsf{C}}{\leftrightarrow} (\epsilon^{AB}\, r^*_B, \epsilon^{\dot{A}\dot{B}} l^*_{\dot{B}}) \quad (x_0, \vec{x})$$

$$(l^A, r^{\dot{A}}) \quad (x_0, \vec{x}) \overset{\mathsf{CP}}{\leftrightarrow} (\delta^A_{\dot{A}} \epsilon^{\dot{A}\dot{B}} l^*_{\dot{B}}, \delta^{\dot{A}}_A \epsilon^{AB} r^*_B) \quad (x_0, -\vec{x})$$

$$(l^A, r^{\dot{A}}) \quad (x_0, \vec{x}) \overset{\mathsf{T}}{\leftrightarrow} (\delta^{A\dot{A}} l^*_{\dot{A}}, \delta^{\dot{A}A} r^*_A) \quad (-x_0, \vec{x})$$

which is inducible on product representations.

5. THE STANDARD MODEL BREAKDOWN OF P AND CP

A relativistic dynamics, characterized by a Lagrangian for the fields involved, may be invariant with respect to an operation group G, e.g., the C, P, and T reflections, or not. A breakdown of the symmetry can occur in two different ways: Either the symmetry is represented on the field value space V, but the Lagrangian is not G-invariant, or there does not even exist a G-representation on V. Both cases occur in the standard model for quark and lepton fields.

5.1. Standard Model Breakdown of P

The charge $\mathbf{U}(1)$ vertex in electrodynamics for a Dirac electron–positron field Ψ interacting with an electromagnetic gauge field Γ_j

$$-\Gamma_j \overline{\Psi} \gamma^j \Psi = -\Gamma_j (l^* \sigma^j l + r^* \check{\sigma}^j r)$$

is invariant under P and T if the fields have the Weyl spinor-induced behavior given above.

In the standard model of leptons [5] with a left-handed isospin doublet field L and a right-handed isospin singlet field r the hypercharge $\mathbf{U}(1)$ and isospin $\mathbf{SU}(2)$ vertex with gauge fields A_j and $\vec{B}_j$, respectively, and internal Pauli matrices $\vec{\tau}$ reads

$$-A_j \left(L^* \sigma^j \frac{\mathbf{1}_2}{2} L + r^* \sigma^j r \right) + \vec{B}_j L^* \sigma^j \frac{\vec{\tau}}{2} L$$

All gauge fields are assumed with the spinor-induced reflection behavior. The P-invariance is broken in two different ways: One component of the lepton isodoublet, e.g., $1 = \frac{1}{2} \frac{(1 - \tau_3)}{2} L \in W^-_L \cong \mathbb{C}^2$, can be used together with the right-handed isosinglet r as a basis of a Dirac space $\Psi \in W^-_L \oplus W_R \cong \mathbb{C}^4$ with a representation of P. This is impossible for the remaining unpaired left-handed field $\frac{1}{2}\frac{(1 + \tau_3)}{2}$ L $\in W_L^+ \cong \mathbb{C}^2$—here P cannot even be defined. However, also for the left–right pair (l, r) the resulting gauge vertex breaks position space reflection P invariance via the familiar neutral weak interactions, induced by a vector field Z_j arising in addition to the **U**(1)-electromagnetic gauge field Γ_j,

$$-\frac{A_j + B_i^3}{2}\, \mathrm{l}^*\sigma^j \mathrm{l} - A_j \mathrm{r}^*\check{\sigma}^j \mathrm{r} = -\Gamma_j \overline{\Psi}\gamma^j \Psi - Z_j \overline{\Psi}\gamma^i \gamma_5 \Psi$$

$$\text{with} \quad \begin{pmatrix}\Gamma_i \\ Z_j\end{pmatrix} = \frac{1}{4}\begin{pmatrix}3 & 1 \\ -1 & 1\end{pmatrix}\begin{pmatrix}A_i \\ B_j^3\end{pmatrix}$$

There is no parameter involved whose vanishing would lead to a P-invariant dynamics.

5.2. GP-Invariance in the Standard Model of Leptons

The CP-reflection induced by the spinor ‘metric’

$$W_L \overset{\mathsf{CP}}{\leftrightarrow} W_R^T, \qquad 1^A \leftrightarrow \delta_{\dot{A}}^A \epsilon^{\dot{A}\dot{B}} \mathrm{l}_{\dot{B}}^*$$

$$W_R \overset{\mathsf{CP}}{\leftrightarrow} W_L^T, \qquad \mathrm{r}^{\dot{A}} \leftrightarrow \delta_A^{\dot{A}} \epsilon^{AB} \mathrm{r}_B^*$$

has to include also a linear reflection of internal operation representation spaces in the case of Weyl spinors with nonabelian internal degrees of freedom.

For isospin **SU**(2)-doublets this reflection is given by the Pauli isospinor reflection discussed above and is denoted as internal reflection by $I = \epsilon$,

$$\begin{array}{ccc} U & \overset{u}{\rightarrow} & U \\ \epsilon\downarrow & & \downarrow\epsilon, \\ U^T & \underset{\check{u}}{\rightarrow} & U^T \end{array} \qquad \begin{array}{l} u \in \mathbf{SU}(2)(\text{isospin}) \\ \psi^a \overset{I}{\leftrightarrow} \epsilon^{ab}\psi_b^*, \quad a, b = 1, 2 \\ -\vec{\tau} = \epsilon^{-1} \circ \vec{\tau}^T \circ \epsilon \end{array}$$

Therewith the linear *GP*-reflection as particle–antiparticle conjugation including nontrivial isospin eigenvalues

$$\mathsf{G} = \mathsf{IC}, \qquad \mathsf{GP} = \mathsf{ICP}$$

reads for left-handed Weyl spinors and isospinors

$$W_L \otimes U \overset{\mathsf{GP}}{\leftrightarrow} W_R^T \otimes U^T, \qquad \mathrm{L}^{Aa} \leftrightarrow \delta_{\dot{A}}^A \epsilon^{\dot{A}\dot{B}} \epsilon^{ab} \mathrm{L}_{\dot{B}b}^*$$

The antilinear *T*-reflection uses the **U**(2)-scalar product

$$U \overset{*}{\leftrightarrow} U^T, \qquad U \times U \to \mathbb{C}, \qquad \langle \psi^a | \psi^b \rangle = \delta^{ab}$$

$$W_L \otimes U \overset{T}{\leftrightarrow} W_R^T \otimes U_T, \qquad \mathrm{L}^{Aa} \leftrightarrow \delta^{AB}\delta^{ab}\mathrm{L}^*_{Bb}$$

The isospin dual coincides with the time dual $U^T = U^*$.

In the product CPT there arises, in the basis chosen, an isospin transformation $\epsilon^{ac}\delta_{cb} \cong e^{i\pi\tau_2/2} \in \mathbf{SU}(2)$,

$$W_L \otimes U \overset{\mathsf{ICPT}}{\leftrightarrow} W_L \otimes U, \qquad \mathrm{L}^{Aa} \leftrightarrow \delta^{A\dot{B}}\,\delta_{\dot{B}B}\epsilon^{ac}\,\delta_{cb}\mathrm{L}^{Bb}$$

decisive to prove the GPT-theorem with

$$\overline{\mathsf{ICPT}} \in \mathbf{SU}(2) \times \mathbf{SL}(\mathbb{C}^2)$$

With the spinor-induced reflection behavior for the gauge fields the standard model for leptons, i.e., with internal hypercharge-isospin action, allows the representation of GP and T with the gauge vertex above being GP- and T-invariant.

5.3. CP-Problems for Quarks

If quark triplets and antitriplets which come with the dual defining $\mathbf{SU}(3)$ representations are included in the standard model, an extended CP-reflection has to employ a linear reflection γ between dual representation spaces of color $\mathbf{SU}(3)$, i.e., an $\mathbf{SU}(3)$-invariant bilinear form of the representation space,

$$\begin{array}{ccc} U & \overset{D(u)}{\rightarrow} & U \\ \gamma\downarrow & & \downarrow\gamma \\ U^T & \underset{\check{D}(u)}{\rightarrow} & U^T \end{array}, \qquad \begin{array}{l} D:\quad \mathbf{SU}(3) \to \mathbf{GL}(U) \quad \text{(color representation)} \\ \gamma^{-1} \circ D(u)^T \circ \gamma = D(u^{-1}) \quad \text{for all } u \in \mathbf{SU}(3) \end{array}$$

The situation for isospin $\mathbf{SU}(2)$ and color $\mathbf{SU}(3)$ is completely different with respect to the existence of such a linear dual isomorphism γ: All irreducible $\mathbf{SU}(2)$-representations $[2T]$ with isospin $T = 0, 1/2, 1, \ldots$ have, up to a scalar factor, a unique invariant bilinear form $\bigvee^{2T}\epsilon$ as product of the spinor 'metric' discussed above.

That is not the case for the color representations. Some representations are linearly self-dual, some are not.

The complex irreducible representations of $\mathbf{SU}(3)$ are characterized by $[N_1, N_2]$ with two integers $N_{1,2} = 0,1,2, \ldots$. They arise from the two fundamental triplet representations, dual to each other and parametrizable with eight Gell-Mann matrices $\vec{\lambda}$:

$$\text{triplet:}\quad [1, 0] = u = e^{i\vec{\gamma}\vec{\lambda}}, \qquad \text{antitriplet:}\quad [0, 1] = \check{u} = u^{-1T} = (e^{-i\vec{\gamma}\vec{\lambda}})^T$$

The $\mathbf{SU}(3)$-representation $[N_1, N_2]$ acts on vector space U with $\dim_{\mathbb{D}} U = \frac{(N_1 + 1)(N_2 + 1)(N_1 + N_2 + 2)}{2}$.

Dual representations have reflected integer values $[N_1, N_2] \leftrightarrow [N_2, N_1]$. Only those $\mathbf{SU}(3)$-representations whose weight diagram is central reflection symmetric in the real 2-dimensional weight vector space (Appendix) have one, and only one, $\mathbf{SU}(3)$-invariant bilinear form[1], i.e., they are linearly self-dual. Dual representations have weights which are reflections of each other,

$$\mathbf{weights}\ [N_1, N_2] \overset{-1_2}{\leftrightarrow} \mathbf{weights}\ [N_2, N_1]$$

Therefore, one obtains as self-dual irreducible $\mathbf{SU}(3)$-representations

$$\mathbf{weights}[N_1, N_2] = -\mathbf{weights}\ [N_1, N_2] \Leftrightarrow N_1 = N_2 = N$$
$$\Rightarrow \dim_{\mathbb{D}} U = (N + 1)^3 = 1, 8, 27, \ldots$$

For example, for the octet $[1, 1]$ as adjoint $\mathbf{SU}(3)$-representation, the Killing form defines its self-duality.

A general remark (Appendix): The Lie group $\mathbf{SL}(\mathbb{C}^{r+1})$ with its maximal compact subgroup $\mathbf{SU}(r + 1)$ of rank r is defined as invariance group of the $\mathbb{C}^{r+1}$-volume elements, which are totally antisymmetric $(r + 1)$-linear forms $\epsilon^{a_1\cdots a_{r+1}}$. Their complex, finite-dimensional, irreducible representations are characterized by r integers $[N_1, \ldots, N_r]$ with the dual representations having the reflected order $[N_r, \ldots, N_1]$. The weights (eigenvalues) for dual representations are related to each other by the central weight space reflection $-\mathbf{1}_r$ which defines the linear particle–antiparticle conjugation I for $\mathbf{SU}(n)$. Only for $n = 2$ [isospin $\mathbf{SU}(2)$] are all representations $[N = 2T]$ self-dual with their invariant bilinear form arising from ϵ^{ab} for [1]. The $n = 2$ self-duality of the doublet $u(2) \cong \check{u}(2)$ is replaced for $n = 3$ by the equivalence of antisymmetric triplet square and antitriplet representation $u(3) \wedge u(3) \cong \check{u}(3)$, i.e., $3 \wedge 3 \cong \overline{3}$, with the obvious generalization $\bigwedge^r u(r + 1) \cong \check{u}(r + 1)$ for general rank r.

Obviously all $\mathbf{SU}(r + 1)$-representations have an invariant sesquilinear form, the $\mathbf{SU}(r + 1)$ scalar product. However, this antilinear structure cannot define a linear particle–antiparticle conjugation.

It is impossible to define a CP-extending duality-induced linear GP-reflection for the irreducible complex 3-dimensional quark representation spaces since there does not exist a color $\mathbf{SU}(3)$-invariant bilinear form of the triplet space $U \cong \mathbb{C}^3$. Or equivalently: There does not exist a (3×3) matrix γ for the reflection $-\vec{\lambda} = \gamma^{-1} \circ \vec{\lambda}^T \circ \gamma$ of all eight Gell-Mann matrices. Therewith there arise also problems to define an $\mathbf{SU}(3)$-compatible time reflection for quark triplet fields. Could all this be the reason for the breakdown of *CP*-invariance in the quark field sector and its parametrization (e.g., Cabibbo–Kobayashi–Maskawa) with three families of color triplets?

APPENDIX. CENTRAL REFLECTIONS OF LIE ALGEBRAS

A representation of a group G on a vector space V is *self-dual* if it is equivalent to its dual representation, defined by the inverse transposed action on the linear forms V^T,

$$\left.\begin{array}{ll} D: & G \to \mathbf{GL}(V) \\ \check{D}: & G \to \mathbf{GL}(V^T) \end{array}\right\}, \qquad \check{D}(g) = D(g^{-1})^T$$

i.e., if the following diagram with a linear or antilinear isomorphism ζ: $V \to V^T$ commutes with the action of all group elements:

$$\begin{array}{ccc} V & \overset{D(g)}{\longrightarrow} & V \\ {\scriptstyle\zeta}\downarrow & & \downarrow{\scriptstyle\zeta}, \\ V^T & \underset{\check{D}(g)}{\longrightarrow} & V^T \end{array} \qquad \zeta^{-1} \circ D(g)^T \circ \zeta = D(g^{-1}) \quad \text{for all } g \in G$$

Self-duality is equivalent to the existence of a nondegenerate bilinear (for linear ζ) or sesquilinear form (for antilinear ζ) of the vector space V,

$$V \times V \to \mathbb{C}, \qquad \zeta(w, v) = \langle \zeta(w), v \rangle$$

$$\text{selfdual} \quad \zeta(g \bullet w, g \bullet v) = \zeta(w, v), \qquad g \bullet v = D(g)(v)$$

For the Lie algebra $L = \log G$ of a Lie group G with dual representations in the endomorphism algebras $\mathbf{AL}(V)$ and $\mathbf{AL}(V^T)$ which are negative transposed to each other

$$\left.\begin{array}{ll} \mathcal{D}: & L \to \mathbf{AL}(V) \\ \check{\mathcal{D}}: & L \to \mathbf{AL}(V^T) \end{array}\right\}, \qquad \check{\mathcal{D}}(l) = -\mathcal{D}(l)^T$$

a self-duality isomorphism, i.e., the reflection $V \overset{\zeta}{\leftrightarrow} V^T$, fulfills

$$\zeta(l \bullet w, v) = -\zeta(w, l \bullet v), \qquad l \bullet v = \mathcal{D}(l)(v)$$

and defines the *central reflection of the Lie algebra* in the representation

$$\begin{array}{ccc} V & \overset{\mathcal{D}(l)}{\longrightarrow} & V \\ {\scriptstyle\zeta}\downarrow & & \downarrow{\scriptstyle\zeta}, \\ V^T & \underset{\check{\mathcal{D}}(l)}{\longrightarrow} & V^T \end{array} \qquad \zeta^{-1} \circ \mathcal{D}(l)^T \circ \zeta = -\mathcal{D}(l) \qquad \text{for all} \quad l \in \log G$$

With Schur's lemma, an irreducible complex finite-dimensional representation of a group or Lie algebra can have at most, up to a constant, one invariant bilinear and one invariant sesquilinear form. For example, Pauli spinors for **SU**(2) have both ϵ^{AB} (bilinear) and δ^{AB} (sesquilinear, scalar product), $A, B = 1, 2$, quark triplets have only a scalar product δ^{ab}, $a, b = 1, 2, 3$, and Weyl spinors for $\mathbf{SL}(\mathbb{C}^2)$ have only the bilinear 'metric' ϵ^{AB}.

For a simple Lie algebra L of rank r, the weights (eigenvalue vectors for a Cartan subalgebra) of dual representations $\mathcal{D}$ and $\check{\mathcal{D}}$ are related to each other by the central reflection of the weight vector space $\mathbb{R}^r$,

$$\mathbf{weights}\ \mathcal{D}[L] \overset{-1_r}{\leftrightarrow} \mathbf{weights}\ \check{\mathcal{D}}[L]$$

which may be induced by a linear isomorphism ζ of the dual representation spaces. Such a linear isomorphism for an L-representation exists [1] if, and only if, the weights of the representation $\mathcal{D}: L \to \mathbf{AL}(V)$ are invariant under central reflection,

$$V \overset{\zeta}{\leftrightarrow} V^T \Leftrightarrow \mathbf{weights}\ \mathcal{D}[L] = -\mathbf{weights}\ \mathcal{D}[L]$$

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